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Ballot number representation of the percolation probability series for the directed square lattice

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Abstract. Series expansion data are matters of increasing importance for studying the directed percolation problem and others which are not yet solved. In order to extrapolate series for the percolation probability on the directed square lattice, Baxter and Guttmann proposed a numerical method based on an assumption that the so-called correction terms are expressed as rational functions of the Catalan numbers. We give a theorem that the coefficients of the series are generally given as finite series of the ballot numbers, which proves the assumption by Baxter and Guttmann as a corollary. The proof of the theorem gives a method to calculate correction terms exactly, as demonstrated by calculating the first three correction terms explicitly. Although the present work provides a mathematical basis for the extrapolation procedure, there are still open problems concerning this procedure.

1. Introduction

Directed percolation (DP) on the square lattice, originally introduced as a simple probabilistic model of a flow of fluid through a random media (Broadbent and Hammersley 1957), can be regarded as a simple model of a spread of influence in $1 + 1$ dimensions (Domany and Kinzel 1984, Kinzel 1985). It has been associated with a wide variety of non-equilibrium lattice models such as the contact process (Harris 1974, Liggett 1985), the branching annihilating random walk (Bramson and Gray 1985, Takayasu and Tretyakov 1992, Jensen 1993) and the ZGB model (Ziff *et al* 1986). It was conjectured that, if a model with a scalar order parameter exhibits a continuous transition into a unique absorbing state, the critical behaviour is generically of the DP type (Janssen 1981, Grassberger 1982). The Reggeon field theory (Grassberger and de la Torre 1979, Cardy and Suger 1980) and the damage spreading transition (Martins *et al* 1991, Grassberger 1995) belong to this DP universality class and there is no counterexample to this conjecture as yet (Dickman 1993, Grassberger 1995).

Although the DP universality class seems to be very wide, no model in it is exactly solved. The most reliable evaluation of critical exponents is given by numerical methods using series expansion data for the original DP model. Baxter and Guttmann (1988) calculated a 41 term series for the bond percolation probability, P, on the directed square lattice. Jensen and Guttmann (1995) have extended the series from 41 terms to 54 terms. It is very interesting to see that the conjecture that the critical exponent for the percolation probability, β, may be exactly $\frac{199}{720}$ given by the former paper is denied by the latter one.

Jensen and Guttmann (1995, 1996a) concluded that the critical exponents for the DP should not be expected to be simple rational fractions. This conjecture is remarkable, since all the exactly solved models in two dimensions have rational exponents. Quite recently Guttmann and Enting (1996) proposed a numerical procedure that indicates whether or not a given statistical mechanical system or a combinatorial problem is solved in the sense of being expressed in terms of D-finite functions. It should be noted that their technique uses the date of series expansions. Their analysis shows that the DP belongs to the 'unsolvable' class (Guttmann and Enting 1996, Guttmann 1996).

Now the series expansion data are matters of increasing importance to the study of the DP and other unsolved models. We should, therefore, notice that Baxter and Guttmann (1988) and Jensen and Guttmann (1995) have extended the series for the percolation probability, P, on the directed square lattice based on an assumption. Here we briefly explain their extrapolation procedure. Let P_n be the finite-lattice approximation for P obtained by a lattice with a linear size n. Baxter and Guttmann (1988) calculated P_n as a power series of q, which is the probability that each bond is closed, up to $n = 29$ and observed that the difference, $P_n - P_{n+1}$, is of the order of q^{n+1}. This observation led them to define the *correction terms* $\{d_{n,l}\}$ as

$$P_n - P_{n+1} = q^n \sum_{l \geqslant 1} d_{n,l} q^l. \tag{1.1}$$

Using finite numerical data $\{d_{n,l}\}$, they estimated the correction terms $d_{n,l}$ as functions of n for $l = 1, 2, \dots, 12$. The expressions are given as linear combinations of the Catalan number

$$c_n = \frac{1}{n+1}\binom{2n}{n} \qquad n = 1, 2, 3, \dots \tag{1.2}$$

in which coefficients are polynomials of n. It was conjectured that the correction terms can be generally expressed as rational functions of the Catalan numbers. They assumed that their expressions for $d_{n,l}$, $l = 1, 2, \dots, 12$, are valid also for $n > 29$ and, using (1.1), they extended the series of P from the 29 terms to $29 + 12 = 41$ terms. Jensen and Guttmann (1995) performed the same procedure to extrapolate the series from the directly calculated 39 terms to 54 terms.

Recently Bousquet-Mélou (1996) proved formula (1.1) and exactly calculated, by using a combinatorial method, the first two correction terms, $d_{n,1}$ and $d_{n,2}$, which are the same as those conjectured by Baxter and Guttmann. Inui and Katori (1996) obtained the same results by another method. There has been, however, no theoretical support so far for the conjecture that $d_{n,l}$ can be generally expressed by rational functions of the Catalan numbers.

In the present paper we generalize the method reported in the previous paper (Inui and Katori 1996) and give a mathematical basis for the extrapolation procedure for the first time. We consider coefficients of the series expansion of the probability $P_{n,m}$, which will be shortly defined, and prove a theorem that they are expressed by a finite series of ballot numbers $\{\alpha_{n,m}\}$ (we call them the *ballot number representation* (BNR)). Since $P_n = \sum_{m=1}^{n} P_{n,m}$ and $c_n = \alpha_{n,1}$, the above mentioned conjecture on $d_{n,l}$ is proved as a corollary of our theorem. Our proof provides not only theoretical support for the extrapolation method, but also a method to calculate $d_{n,l}$ exactly.

Baxter and Guttmann (1988) and Jensen and Guttmann (1995) reported many good properties concerning the number of terms and the coefficients for the Catalan number representations observed in their numerical data. Although the reason why the correction terms can be expressed using the Catalan numbers is clarified by the present theorem, still we cannot explain these additional properties. We will also discuss these open problems from the view point of the BNR.

The paper is organized as follows. In section 2, we define the probability $P_{n,m}$, which is expressed as a finite series. It is shown that the coefficients $\{a_{n,m}^{(s)}\}_{s=0,1,2,\ldots}$ in the series are given as numbers of bond configurations on a finite lattice which satisfy some conditions. According to the *cluster number* c, which characterizes bond configurations, $a_{n,m}^{(s)}$ is classified as $a_{n,m}^{(s)} = \sum_{c\geqslant 1} a_{n,m,c}^{(s)}$. We show that $b_{n,m}^{(s)} \equiv \sum_{c\geqslant 2} a_{n,m,c}^{(s)}$ can be expressed using $\{a_{n',m',1}^{(s')}\}$ with $n' < n$ and $s' < s$. It is noted that $\{a_{n,m,1}^{(s)}\}$ are decoupled from $\{b_{n,m}^{(s)}\}$ and they are given as solutions of difference equations. In section 3, the difference equations are generally solved and a concept of the BNR is introduced to characterize the solutions. By mathematical induction with respect to s, we prove that the series $\{a_{n,m,1}^{(s)}\}$ has the BNR for any $s \geqslant 0$. Hence, we conclude the main theorem that the series of coefficients $\{a_{n,m}^{(s)}\}$ has the BNR for any $s \geqslant 0$. In section 4, we exactly calculate $\{a_{n,m}^{(s)}\}$ for $s = 1$ and 2 and give their explicit representations. In section 5, we discuss the relation between the coefficients $\{a_{n,m}^{(s)}\}$ and the correction terms $\{d_{n,l}\}$. Future problems are given in section 6. In order to prove theorems and perform exact calculations, we need many combinatorial identities associated with the ballot numbers. The derivations of them are given in appendices.

2. Difference equations for coefficients

We consider a down-pointing triangular region in the square lattice with a linear size n,

$$V_n^0 = \{(x, y) \in \mathbb{Z}^2 : x + y = \text{even},\ 0 \leqslant y \leqslant n - 1,\ -y \leqslant x \leqslant y\} \tag{2.1}$$

in which we assume that there is a bond between each pair of nearest-neighbour sites. V_n^0 has $n(n-1)$ bonds. We assume that each bond is either open with probability p or closed with probability $q = 1 - p$. We say *there is an open path from* (x_0, y_0) *to* $(x_r, y_0 + r)$ for $r \geqslant 1$, if there is a sequence $(x_0, y_0), (x_1, y_0 + 1), \ldots, (x_r, y_0 + r)$ of sites in V_n^0 such that for each $0 \leqslant k \leqslant r - 1$ the bond from $(x_k, y_0 + k)$ to $(x_{k+1}, y_0 + k + 1)$ is open. We regard two sites as connected if there is at least one open path between them. Let $P_{n,m}$ be the probability that the origin (0,0) is connected to exactly m sites on the top row $\bar{V}_n^0 = \{(x, n-1) \in V_n^0 : -(n-1) \leqslant x \leqslant n-1\}$. In our previous paper (Inui and Katori 1996) we proved that it is given in the form,

$$P_{n,m} = p^{n^2-2n+m} q^{n-m} \sum_{s=0}^{(n-1)(n-2)} a_{n,m}^{(s)} p^{-s} q^s \tag{2.2}$$

where $a_{n,m}^{(s)}$ is the number of bond configurations on V_n^0 such that exactly $n - m + s$ bonds are closed and (0,0) is connected to exactly m sites in $\bar{V}_n^0$. Here the statement that exactly $n - m + s$ bonds are closed means that other $n^2 - 2n + m - s$ bonds are open. If we define $P_n = \sum_{m=1}^{n} P_{n,m}$, then P_n is a finite-lattice approximation for the percolation probability P.

In order to give equations for $a_{n,m}^{(s)}$, we introduce the following notations. For $1 \leqslant n_1 < n_2$, let a trapezium $V_{[n_1,n_2]}^0 = V_{n_2}^0 \setminus V_{n_1-1}^0$ and B_{n_1,n_2} be the set of all bonds between the nearest-neighbour pairs of sites which are both in $V_{[n_1,n_2]}^0$. We define $\mathcal{B}_{n_1,n_2}$ as the set of all bond configurations on B_{n_1,n_2}. The total number of bond configurations in $\mathcal{B}_{n_1,n_2}$ is $2^{n_2(n_2-1)-n_1(n_1-1)}$. We consider some special bond configurations which satisfy a condition, $\mathcal{C}$, and we write their total number as $\sharp\{\omega \in \mathcal{B}_{n_1,n_2} : \mathcal{C}\}$. Let $\Lambda_n = \{-(n-1), -(n-1)+2, \ldots, (n-1)-2, (n-1)\}$ and Y_n be the set of all subsets of Λ_n. For a set $A \in Y_n$, the cardinality $|A|$ denotes the number of sites included in A. If $x, y \in A$ and $|x - y| = 2$, we say that x and y are adjacent, a sequence of adjacent sites is called a *cluster*. We write the number of clusters in A as $c(A)$. For a given bond configuration $\omega \in \mathcal{B}_{n_1,n_2}$ and a non-empty set $B \in Y_{n_1}$, a set of sites in Y_n which are connected with at least one site in B is determined for each $n \in \{n_1, n_1+1, \ldots, n_2\}$. We write it as $A_n(A_{n_1} = B; \omega)$. Using these notations, the value $a^{(s)}_{n,m}$ is given as $a^{(s)}_{n,m} = \sum_{c\geqslant 1} a^{(s)}_{n,m,c}$ with $a^{(s)}_{n,m,c} = \sum_{C\in Y_n} 1_{\{|C|=m, c(C)=c\}} \sharp\{\omega \in \mathcal{B}_{1,n}$: exactly $n-m+s$ bonds are closed and $A_n(A_1 = \{0\}; \omega) = C\}$. Here $1_{\{\Omega\}}$ is the indicator function of an event, Ω, such that $1_{\{\Omega\}} = 1$ if Ω is satisfied and $1_{\{\Omega\}} = 0$ otherwise.

For $C_1 \in Y_{n_1}, C_2 \in Y_{n_2}$, define

$$\begin{aligned} F(A_{n_2} = C_2, A_{n_1} = C_1; \Delta s) = \sharp\{\omega \in \mathcal{B}_{n_1,n_2} :\ & \text{exactly } (n_2 - n_1) - (|C_2| - |C_1|) \\ & + \Delta s \text{ bonds are closed, } A_{n_2}(A_{n_1} = C_1; \omega) = C_2, \\ & \text{and } c(A_n(A_{n_1} = C_1; \omega)) \geqslant 2\ \forall n \in \{n_1+1, \ldots, n_2-1\}\}. \end{aligned} \tag{2.3}$$

We define

$$\begin{aligned} f((n_2, m_2), (n_1, m_1); \Delta s) = & \sum_{C_1\in Y_{n_1}} \sum_{C_2 \in Y_{n_2}} 1_{\{|C_1|=m_1, c(C_1)=1\}} 1_{\{|C_2|=m_2, c(C_2)=1\}} \\ & \times F(A_{n_2} = C_2, A_{n_1} = C_1; \Delta s) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} g((n_2, m_2), (n_1, m_1); \Delta s) = & \sum_{C_1\in Y_{n_1}} \sum_{C_2 \in Y_{n_2}} 1_{\{|C_1|=m_1, c(C_1)=1\}} 1_{\{|C_2|=m_2, c(C_2)\geqslant 2\}} \\ & \times F(A_{n_2} = C_2, A_{n_1} = C_1; \Delta s). \end{aligned} \tag{2.5}$$

We introduce a difference operator for a double series $\{\beta_{n,m}\}$ as

$$D(\beta_{n,m}) = \beta_{n+1,m} - (\beta_{n,m-1} + 2\beta_{n,m} + \beta_{n,m+1}). \tag{2.6}$$

We have the following lemma.

Lemma 2.1. Let

$$a^{(s)}_{n,m} = a^{(s)}_{n,m,1} + b^{(s)}_{n,m} \qquad \text{with } b^{(s)}_{n,m} = \sum_{c\geqslant 2} a^{(s)}_{n,m,c} \qquad \text{for } s \geqslant 0. \tag{2.7}$$

(i) If $m > n$, $a^{(s)}_{n,m} = a^{(s)}_{n,m,1} = b^{(s)}_{n,m} = 0$.
(ii) For $1 \leqslant m \leqslant n$,

$$D(a^{(0)}_{n,m,1}) = 0 \qquad \text{and} \qquad b^{(0)}_{n,m} = 0. \tag{2.8}$$

(iii) When $1 \leqslant s \leqslant n$ and $1 \leqslant m \leqslant n$,

$$D(a^{(s)}_{n,m,1}) = \sum_{k=0}^{s-1} \sum_{s'=0}^{s-1-k} \sum_{m'} f((n+1, m), (n-k, m'); s-s') a^{(s')}_{n-k,m',1} \tag{2.9}$$

and

$$b^{(s)}_{n,m} = \sum_{k=0}^{s-1} \sum_{s'=0}^{s-1-k} \sum_{m'} g((n, m), (n-k-1, m'); s-s') a^{(s')}_{n-k-1,m',1}. \tag{2.10}$$

Proof. By definition, $P_{n,m} = 0$ for $m > n$. Since $a^{(s)}_{n,m,c} \geqslant 0$, and $a^{(s)}_{n,m} = \sum_{c\geqslant 1} a^{(s)}_{n,m,c}$, (i) is concluded by (2.2). We have remarked that $a^{(s)}_{n+1,m,1}$ is given as a number of bond configurations $\{\omega\}$ in which exactly $(n+1) - m + s$ bonds are closed and $|A_{n+1}(A_1 = \{0\}; \omega)| = m$, $c(A_{n+1}(A_1 = \{0\}; \omega)) = 1$. In figure 1, we show three typical examples of such bond configurations. Only open bonds and sites which are connected to $\{0\}$ (marked by full circles) are shown. We regard these marked sites as occupied sites and others as

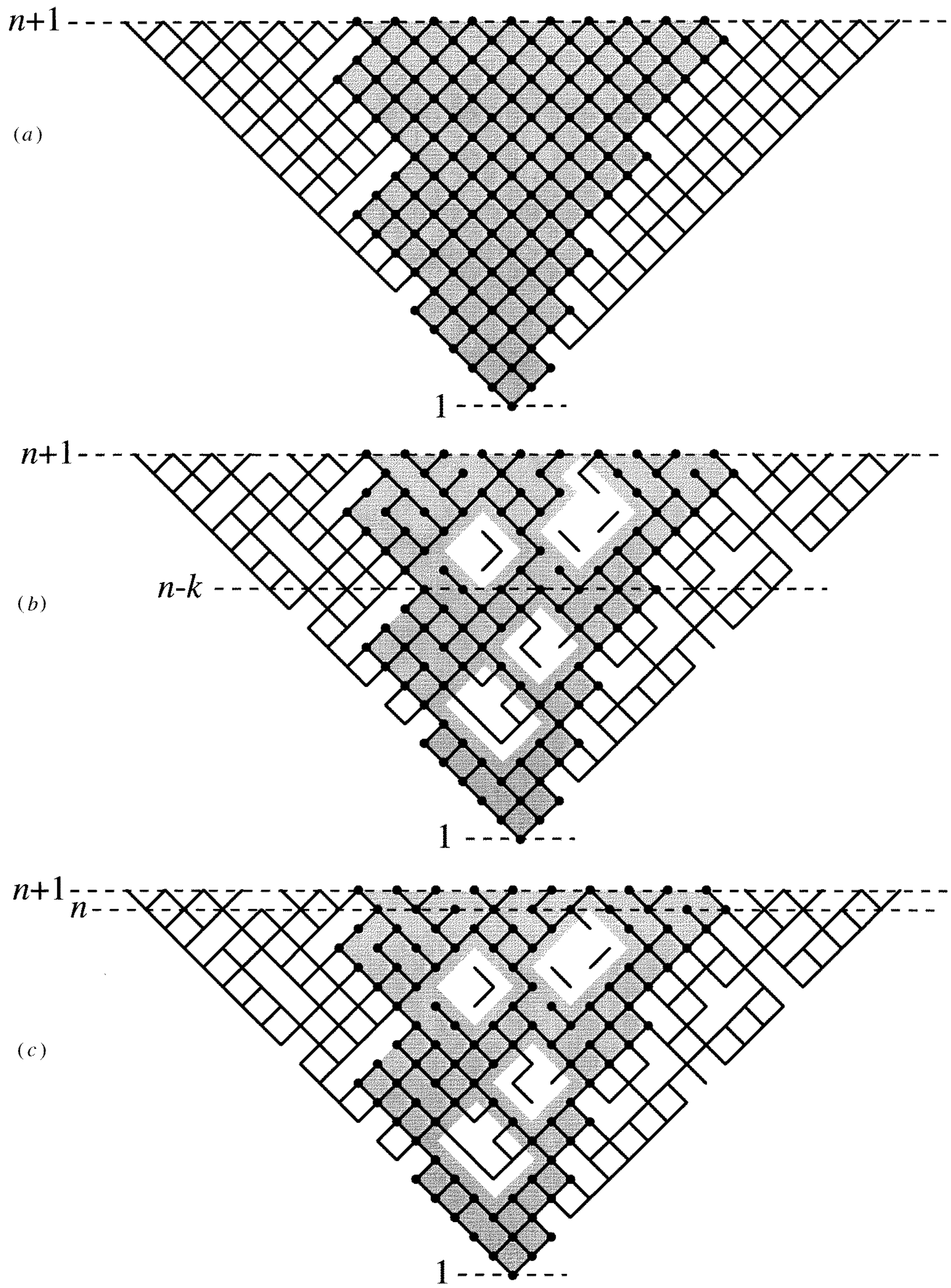

Figure 1. Three typical examples of bond configurations. Shaded regions indicate animals.

vacant sites. Here we call the set of occupied sites and open bonds between them an animal. Figure 1(*a*) shows a case where there are no *holes* (clumps of vacant sites) nor closed bonds inside of the animal, such an animal is said to be *compact*. Bousquet-Mélou (1996) proved that $a^{(0)}_{n+1,m} = a^{(0)}_{n+1,m,1}$ = the number of compact animals of directed height $n+1$ and width m at $y = n$. If we use this identification, it is easy to confirm (ii) (Inui and Katori 1996). When $s \geqslant 1$, $a^{(s)}_{n+1,m,1}$ is the number of non-compact animals with appropriate conditions. Figure 1(*b*) shows the case where the animal has four holes and many closed bonds in it. We assume that there is a positive integer, k, such that

$$c(A_t(A_1 = \{0\}; \omega)) \geqslant 2 \qquad \text{for } n-k+1 \leqslant t \leqslant n \tag{2.11}$$

and

$$c(A_{n-k}(A_1 = \{0\}; \omega)) = 1 \qquad |A_{n-k}(A_1 = \{0\}; \omega)| = m'. \tag{2.12}$$

Assume that exactly $\{(n+1)-(n-k)\}-(m-m')+(s-s')$ bonds are closed in $V^0_{[n-k,n+1]}$. It follows that exactly $(n-k)-m'+s'$ bonds are closed in V^0_{n-k}. Since we consider the case (2.11), $s-s' \geqslant \{(n+1)-(n-k)\} = k+1$. Figure 1(*c*) shows the case in which $c(A_n(A_1 = \{0\}; \omega)) = 1$. Counting the number of such bond configurations gives $a^{(s)}_{n+1,m,1}$ for $s \geqslant 1$. From (2.3) and (2.4), we have

$$\begin{aligned} a^{(s)}_{n+1,m,1} = & \sum_{s'} \sum_{k \geqslant 1} \sum_{m'} 1_{\{s-s' \geqslant k+1\}} f((n+1,m),(n-k,m');s-s') a^{(s')}_{n-k,m',1} \\ & + \sum_{s'} \sum_{m'} f((n+1,m),(n,m');s-s') a^{(s')}_{n,m',1} \end{aligned} \tag{2.13}$$

where the bond configurations such as figure 1(*b*) (resp. (*c*)) contribute to the first (resp. second) summation on the RHS. Since $f((n+1,m),(n-k,m');s-s') = 0$ if $s-s' < 0$ and $a^{(s')}_{n-k,m',1} = 0$ if $s' < 0$, (2.13) is written as

$$\begin{aligned} a^{(s)}_{n+1,m,1} = & \sum_{s'=0}^{s} \sum_{k \geqslant 1} \sum_{m'} 1_{\{s-s' \geqslant k+1\}} f((n+1,m),(n-k,m');s-s') a^{(s')}_{n-k,m',1} \\ & + \sum_{s'=0}^{s} \sum_{m'} f((n+1,m),(n,m');s-s') a^{(s')}_{n,m',1} \\ = & \sum_{m'} f((n+1,m),(n,m');0) a^{(s)}_{n,m',1} \\ & + \sum_{s'=0}^{s-1} \sum_{k=0}^{s-s'-1} \sum_{m'} f((n+1,m),(n-k,m');s-s') a^{(s')}_{n-k,m',1}. \end{aligned} \tag{2.14}$$

We can see that (Inui and Katori 1996)

$$f((n+1,m),(n,m');0) = \begin{cases} 1 & \text{if } m' = m-1 \text{ or } m+1 \\ 2 & \text{if } m' = m \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

and we obtain the first equation of (iii). The second equation of (iii) is also derived by the same argument. □

The functions $f((n+1,m),(n-k,m');s-s')$ and $g((n,m),(n-k-1,m');s-s')$ are polynomials with respect to n. Appendix A proves the following lemma concerning the degrees of these polynomials.

Lemma 2.2. The functions $f((n+1,m),(n-k,m');s-s')$ and $g((n,m),(n-k-1,m');s-s')$ are polynomials of n for m, k, m' and $s-s'$. Their degrees are at most $s-s'$.

3. Ballot number representation

We introduce the following numbers with three indices.

$$\alpha_{n,m,k} = \binom{2(n-1)}{(n-1)+m-k} - \binom{2(n-1)}{(n-1)+m+k}. \tag{3.1}$$

As usual we assume that $\binom{N}{M} = 0$ if $M < 0$ or $M > N$. In particular,

$$\alpha_{n,m} \equiv \alpha_{n,m,1} = \binom{2(n-1)}{(n-1)+m-1} - \binom{2(n-1)}{(n-1)+m+1}. \tag{3.2}$$

Remark. The number defined by

$$b_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1} = \frac{n+1-m}{n+1}\binom{n+m}{m} \tag{3.3}$$

is called a *ballot number* (Riordan 1979). Consider a ballot in which candidate A scores α votes and candidate B scores β votes with $\alpha > \beta$. The probability that, during the ballot, A was always ahead of B is given by $b_{\alpha-1,\beta}/\binom{\alpha+\beta}{\alpha} = (\alpha-\beta)/(\alpha+\beta)$ (see the Ballot theorem, for example, in Grimmett and Stirzaker 1992 p 77). We find that $\alpha_{n,m} = b_{n+m-1,n-m}$, in this present paper will simply call it the ballot number.

By equation (3.2), we find the following basic properties of the ballot number.

$$\alpha_{n,-m} = -\alpha_{n,m} \tag{3.4}$$

$$\alpha_{n,m} = 0 \qquad \text{if } |m| > n \tag{3.5}$$

and

$$D(\alpha_{n,m}) = 0 \tag{3.6}$$

where $D(\cdot)$ is the difference operator defined by (2.6).

As we put $m = 1$ in (3.2), it got reduced to the Catalan number,

$$c_n = \alpha_{n,1} = \frac{1}{n+1}\binom{2n}{n} \tag{3.7}$$

which appears in many combinatorial problems (Sloane 1973).

As shown in appendix B, we can give the following representation for double series $\{\beta_{n,m}\}$.

Lemma 3.1. If $\beta_{n,m} = 0$ for $m > n$, then with a given $n_0 \geqslant 1$

$$\beta_{n,m} = \sum_{t=-n_0+1}^{n_0-1} \alpha_{n-n_0+1,m+t}\beta_{n_0,1+|t|} + \sum_{l=1}^{n-n_0}\sum_{w\geqslant 1} \alpha_{n-(l+n_0-1),m,w} D(\beta_{l+n_0-1,w}) \tag{3.8}$$

for $n \geqslant n_0$ and $1 \leqslant m \leqslant n$.

Definition 3.2. Let $\mathcal{A}(n; d)$ be the set of linear combinations of the ballot numbers $\{\alpha_{n,m'}\}$ in the form

$$\sum_{m'=1}^{n} C_{m'}(n)\alpha_{n,m'} \tag{3.9}$$

where $\{C_{m'}(n)\}$ are polynomials of n of at most degree d;

$$C_{m'}(n) = \sum_{r=0}^{d} C_{m',r} n^r. \tag{3.10}$$

If $\beta_n \in \mathcal{A}(n-n_0; d) \quad \forall n \geqslant n_0$, in which n_0 and d are independent of n, we say that the series $\{\beta_n\}$ has the $\mathcal{A}(n-n_0; d)$-BNR.

We prove the following lemma in appendix C.

Lemma 3.3. Let $(l)_0 = 1$ and $(l)_k = l(l+1)(l+2)\dots(l+k-1)$ for $k \geqslant 1$. For each $m, \gamma, \delta \geqslant 1$ and $k \geqslant 0$,

$$\sum_{l=1}^{n-1} \alpha_{n-l,m,\gamma}(l)_k \alpha_{l,\delta} \in \mathcal{A}(n;k). \tag{3.11}$$

Combining lemmas 3.1 and 3.3 gives the following.

Proposition 3.4. Assume that $\beta_{n,m} = 0$ for $m > n$. For each m, if $D(\beta_{n,m}) \in \mathcal{A}(n-n_0+1;d)$ with $n_0 \geqslant 1$, then $\beta_{n,m} \in \mathcal{A}(n-n_0+1;d)$.

Now we prove the main theorem in the present paper.

Theorem 3.5. For any $s \geqslant 0$, $\{a^{(s)}_{n,m}\}$ has the $\mathcal{A}(n-s;s)$-BNR for each m.

Proof. This theorem can be proved by mathematical induction with respect to s. First we apply lemma 3.1 to $\beta_{n,m} = a^{(0)}_{n,m,1}$ with $n_0 = 1$. By lemma 2.1(i) and (ii), $\alpha^{(0)}_{1,1+|t|,1} = \delta_{t,0}$, where $\delta_{i,j}$ denotes Kronecker's delta, and thus $a^{(0)}_{n,m,1} = a^{(0)}_{n,m} = \alpha_{n,m}$ for $1 \leqslant m \leqslant n$. This means that $a^{(0)}_{n,m} \in \mathcal{A}(n;0)$. Assume that $a^{(s')}_{n,m',1} \in \mathcal{A}(n-s';s')$ for $0 \leqslant s' \leqslant s-1$ and $1 \leqslant m' \leqslant n$. From lemmas 2.1(iii) and 2.2, it follows that $D(a^{(s)}_{n,m,1}) \in \mathcal{A}(n-s+1;s)$ and $b^{(s)}_{n,m} \in \mathcal{A}(n-s;s)$. Proposition 3.4 guarantees that $a^{(s)}_{n,m,1} \in \mathcal{A}(n-s+1;s) \subset \mathcal{A}(n-s;s)$ and thus $a^{(s)}_{n,m} = a^{(s)}_{n,m,1} + b^{(s)}_{n,m} \in \mathcal{A}(n-s;s)$. This completes the proof. □

4. Exact calculations for $s = 1$ and 2

In section 3, we introduced a concept of the $\mathcal{A}(n-n_0;d)$-BNR for series $\{\beta_n\}$. In definition 3.2, it should be noted that n_0 and d are independent of n, while the range of summation with respect to m' in (3.9) depends on n. This should be generalized as follows for double series $\{\beta_{n,m}\}$.

Definition 4.1. Let $\mathcal{A}(n,[m-t_0,m+t_1];d)$ be the set of linear combinations of the ballot numbers $\{\alpha_{n,m'}\}$ in the form

$$\sum_{m'=m-t_0}^{m+t_1} C_{m,m'}(n)\alpha_{n,m'} \tag{4.1}$$

where $\{C_{m,m'}(n)\}$ are the polynomials of n of at most degree d;

$$C_{m,m'}(n) = \sum_{r=0}^{d} C_{m,m',r} n^r. \tag{4.2}$$

If $\beta_{n,m} \in \mathcal{A}(n-n_0,[m-t_0,m+t_1];d)$ $\forall n \geqslant n_0$, for each m, in which n_0, t_0, t_1, d are independent of n, we say that the double series $\{\beta_{n,m}\}$ has the $\mathcal{A}(n-n_0;[m-t_0,m+t_1];d)$-BNR.

Theorem 3.5 guarantees that $\{a^{(1)}_{n,m}\}$ and $\{a^{(2)}_{n,m}\}$ have the $\mathcal{A}(n-1;1)$ and $\mathcal{A}(n-2;2)$-BNR for each m. Exact calculation gives, however, the following remarkable results.

Theorem 4.2.

(i) $\{a^{(1)}_{n,m}\}$ has the $\mathcal{A}(n,[m,m+1];2)$-BNR.

(ii) $\{a^{(2)}_{n,m}\}$ has the $\mathcal{A}(n-1,[m-1,m+3];4)$-BNR.

In this section, we demonstrate how these results are obtained using the formulae (2.9) and (2.10) given in lemma 2.1 and lemma 3.1.

Before that, we give here an additional lemma. Although this is equivalent to the fact that $a^{(s)}_{n,m} = 0$ for $s > (n-1)(n-2)$ as shown in (2.2), it will be useful to treat the first term of (3.8).

Lemma 4.3. Assume that $s \geqslant 1$.

$$a^{(s)}_{n,m} = 0 \qquad \text{if } n \leqslant \lfloor \tfrac{1}{2}(3+\sqrt{4s-3})\rfloor \tag{4.3}$$

where $\lfloor N \rfloor$ denotes the largest integer not greater than N.

4.1. $s = 1$

For $s = 1$ formula (2.9) with (2.4) and (2.5) gives the follows. When $1 \leqslant m \leqslant n$,

$$\begin{aligned} D(a^{(1)}_{n,m,1}) &= \sum_{m'} f((n+1,m),(n,m');1)a^{(0)}_{n,m',1} \\ &= 2(n-1)\alpha_{n,m-1} + 4(n-1)\alpha_{n,m} + 2n\alpha_{n,m+1} + 2\alpha_{n,m+2} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} b^{(1)}_{n,m} &= \sum_{m'} g((n,m),(n-1,m');1)a^{(0)}_{n-1,m',1} \\ &= (m-1)(\alpha_{n-1,m} + 2\alpha_{n-1,m+1} + \alpha_{n-1,m+2}) \end{aligned} \tag{4.5}$$

where we have used the fact that $\alpha^{(0)}_{n,m,1} = \alpha_{n,m}$. Hence, lemma 3.1 with $n_0 = 1$ gives

$$\begin{aligned} a^{(1)}_{n,m,1} = {} & 2\sum_{l=1}^{n-1}\sum_{w\geqslant 1}\alpha_{n-l,m,w}(l-1)\alpha_{l,w-1} + 4\sum_{l=1}^{n-1}\sum_{w\geqslant 1}\alpha_{n-l,m,w}(l-1)\alpha_{l,w} \\ & +2\sum_{l=1}^{n-1}\sum_{w\geqslant 1}\alpha_{n-l,m,w}l\alpha_{l,w+1} + 2\sum_{l=1}^{n-1}\sum_{w\geqslant 1}\alpha_{n-l,m,w}\alpha_{l,w+2} \end{aligned} \tag{4.6}$$

since $a^{(1)}_{1,m,1} = 0$ by lemma 4.3. If we apply lemma 3.3, or more explicitly, lemma C.3 (given in appendix C), we will obtain the $\mathcal{A}(n;1)$-BNR for $\alpha^{(1)}_{n,m,1}$. In this procedure we perform summations with respect to l, but do not calculate the summations for w in (4.6). The following formulae for double summations are available, which are derived in appendix D. Note that $\alpha_{n,m,0} = 0$ by definition.

Lemma 4.4.

(i) For $n \geqslant 2$, $m \geqslant 1$ and $k = 0$ and 1,

$$\begin{aligned} (k+1)\sum_{l=1}^{n}\sum_{w=-t+1}^{\infty}\alpha_{n+1-l,m,w}(l)_k\alpha_{l,w+t} = {} & -\sum_{l=1}^{n-1}\alpha_{n-l,m,t}(l)_{k+1}\alpha_{l,1} \\ & +(n)_{k+1} \times \begin{cases} \alpha_{n,m+t} & \text{if } t \geqslant 0,\ m \geqslant t \\ \alpha_{n,m+t} + \alpha_{n,m-t} & \text{if } t \geqslant 2,\ m \leqslant t-1 \\ \alpha_{n,m+t} & \text{if } t \leqslant -1,\ m \geqslant -t+1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.7}$$

(ii) If $m - t \geqslant 0$ and $t \geqslant 1$ or $m - t \leqslant 0$ and $t \leqslant -1$,

$$\sum_{l=1}^{n-1}\alpha_{n-l,m,t}(l)_1\alpha_{l,1} = \mathrm{sgn}(t)\sum_{k=0}^{|t|-1}\frac{(n)_1}{|m-t|+2k+2}\alpha_{n,|m-t|+2k+2} \tag{4.8}$$

and

$$\sum_{l=1}^{n-1} \alpha_{n-l,m,t}(l)_2 \alpha_{l,1} = \mathrm{sgn}(t) \left\{ \sum_{k=0}^{|t|-1} \frac{(n)_2}{|m-t|+2k+2} \alpha_{n,|m-t|+2k+2} - \sum_{k=0}^{|t|-1} \sum_{p \geqslant 0} \frac{(|m-t|+2k+1)(n+1)_1}{|m-t|+2(k+p)+3} \alpha_{n+1,|m-t|+2(k+p)+3} \right\} \tag{4.9}$$

where $\mathrm{sgn}(t) = t/|t|$.

Double summations are performed and (4.6) is now written as

$$a^{(1)}_{n,m,1} = (n-1)(n-2)\alpha_{n-1,m-1} + 2(n-1)\left\{n-2-\frac{1}{m}\right\}\alpha_{n-1,m} + (n-1)\left\{n-\frac{2}{m+1}\right\}\alpha_{n-1,m+1} + 2(n-1)\left\{1-\frac{1}{m+2}\right\}\alpha_{n-1,m+2} + 2\alpha_{n,m+1}. \tag{4.10}$$

It gives an $\mathcal{A}(n-1, [m-1, m+1]; 2)$-BNR for $a^{(1)}_{n,m,1}$. We find that, however, it can be simplified as follows by the definition of the ballot number (3.2).

$$a^{(1)}_{n,m,1} = (n^2-2n-m+2)\alpha_{n,m} - 2m\alpha_{n,m+1}. \tag{4.11}$$

Since (3.6) holds, (4.5) is also simplified as

$$b^{(1)}_{n,m} = (m-1)\alpha_{n,m+1} \tag{4.12}$$

and we have the compact expression

$$a^{(1)}_{n,m} = (n^2-2n-m+2)\alpha_{n,m} - (m+1)\alpha_{n,m+1} \tag{4.13}$$

which means (i) of theorem 4.2. Although this final result (4.13) was already given in Inui and Katori (1996), the derivation shown here is more transparent.

4.2. $s = 2$

In the same way as for (4.4) and (4.5), lemma 2.1(iii) gives

$$D(a^{(2)}_{n,m,1}) = \sum_{k=0}^{1} \sum_{s'=0}^{1-k} \sum_{m'} f((n+1,m),(n-k,m'); s-s') a^{(s')}_{n-k,m',1} \tag{4.14}$$

and

$$b^{(2)}_{n,m} = \sum_{k=0}^{1} \sum_{s'=0}^{1-k} \sum_{m'} g((n,m),(n-k-1,m'); s-s') a^{(s')}_{n-k-1,m',1} \tag{4.15}$$

for $n \geqslant 2$, where the functions $f(\cdot)$ and $g(\cdot)$ are given in table 1.

Lemma 3.1 is applied with $n_0 = 2$, where $a^{(2)}_{2,m,1} = 0$ by lemma 4.3. Using double summation formulae such as lemma 4.4, we obtain the result

$$a^{(2)}_{n,m} = \sum_{m'=m-1}^{m+3} C_{m,m'}(n)\alpha_{n-1,m'} \tag{4.16}$$

where the coefficients $\{C_{m,m'}(n)\}$ are given in table 2. Since these coefficients are polynomials of n of at most degree 4, theorem 4.2(ii) is concluded.

Table 1. Functions $f(\cdot)$ and $g(\cdot)$. For $m = 1$, $g((n, m), (n', m'); \Delta s) \equiv 0$ by definition.

m'	$f((n+1, m), (n, m'); 2)$	$f((n+1, m), (n, m'); 1)$	$f((n+1, m), (n-1, m'); 2)$
$m-2 \geqslant 1$	0	0	$m-3$
$m-1 \geqslant 1$	$2n^2 - 5n - m + 5$	$2(n-1)$	$2(2m-5) + 2\delta_{m,2}$
m	$2(n^2 - 5n - m + 4)$	$4(n-1)$	$2(3m-5) + 4\delta_{m,1} + \delta_{m,2}$
$m+1$	$2n^2 - n - m - 5$	$2n$	$2(2m+1) - 2\delta_{m,1}$
$m+2$	$2(2n-3)$	2	$m + 11 - 2\delta_{m,1}$
$m+3$	3	0	8
$m+4$	0	0	2
otherwise	0	0	0
m'	$g((n, m), (n-1, m'); 2)$	$g((n, m), (n-1, m'); 1)$	$g((n, m), (n-2, m'); 2)$ $(m \geqslant 2)$
$m-1 \geqslant 1$	0	0	$2(m-2)$
m	$2(m-1)(n-3)$	$m-1$	$3(3m-5)$
$m+1$	$(m-1)(4n + \frac{1}{2}m - 12)$	$2(m-1)$	$2(8m-11)$
$m+2$	$(m-1)(2n + m - 4)$	$m-1$	$2(7m-8)$
$m+3$	$\frac{1}{2}(m-1)(m+4)$	0	$6(m-1)$
$m+4$	0	0	$m-1$
otherwise	0	0	0

Table 2. Coefficients $\{C_{m,m'}(n)\}$ of the BNR for $a^{(2)}_{n,m}$.

m'	$C_{m,m'}(n)$
$m-1$	$\frac{1}{2}n^4 - 2n^3 - \frac{1}{2}(2m-7)n^2 + (4m-3)n - \frac{1}{2}(3m^2 + 5m - 6)$
m	$n^4 - 4n^3 - 3(m-2)n^2 + 2(3m-2)n - (4m^2 - 7m + 3)$
$m+1$	$\frac{1}{2}n^4 - 2n^3 - \frac{3}{2}(2m-1)n^2 + (4m+1)n - (3m^2 + m - 3) + \delta_{m,1}$
$m+2$	$-(m+1)n^2 + 2n + (3m+1)$
$m+3$	$\frac{1}{2}m^2 + \frac{3}{2}m$

5. Correction terms

In our previous paper (Inui and Katori 1996), we discussed the relation between the coefficients $\{a^{(s)}_{n,m}\}$ and the correction terms $\{d_{n,l}\}$. The result is the following. Let

$$\tilde{a}^{(s)}_{n,m} = \sum_{k=0}^{s} (-1)^{s-k} \binom{n^2 - 2n + m - k}{s-k} a^{(k)}_{n,m}. \tag{5.1}$$

Then

$$d_{n,l} = \sum_{m=1}^{l} \tilde{a}^{(l-m)}_{n,m}. \tag{5.2}$$

As a corollary of theorem 3.5, we have the following.

Corollary 5.1.

(i) For any $s \geqslant 0$, $\{\tilde{a}^{(s)}_{n,m}\}$ has the $\mathcal{A}(n-s; 2s)$-BNR for each $1 \leqslant m \leqslant n$.

(ii)

$$d_{n,l} = \sum_{k=1}^{n-l+1} f^{(l)}_k(n) c_{n-l+k} \tag{5.3}$$

where $\{f_k^{(l)}(n)\}$ are the polynomials of n of at most degree $2(l-1)$.

In order to derive (ii), we used the formula

$$\alpha_{n,m} = \sum_{k=1}^{m} (-1)^{m-k} \binom{k+m-1}{2k-1} c_{n+k-1} \tag{5.4}$$

which can be easily proved (Riordan 1979).

Owing to the factor $(-1)^{s-k}$ in (5.1), however, cancellation of terms occurs and the degree of polynomials $\tilde{a}_{n,m}^{(s)}$ can be reduced. From (4.13) and (4.16), we obtain

$$\begin{aligned}
\tilde{a}_{n,m}^{(0)} &= \alpha_{n,m} \\
\tilde{a}_{n,m}^{(1)} &= -2(m-1)\alpha_{n,m} - (m+1)\alpha_{n,m+1} \\
\tilde{a}_{n,m}^{(2)} &= (2mn - 6m + 5 - \delta_{n,1}\delta_{m,1})\alpha_{n,m} - 2mn\alpha_{n,m+1} + \tfrac{1}{2}m(m+3)\alpha_{n,m+2} \\
&\quad -2\alpha_{n-1,m} - (2-\delta_{m,1})\alpha_{n-1,m+1}
\end{aligned} \tag{5.5}$$

for $1 \leqslant m \leqslant n$. There is further cancellation of terms, when we perform summation (5.2), since $\tilde{a}_{n,m}^{(s)}$ can be positive and negative as shown above. The final results for correction terms, which can be regarded as the corollary of theorem 4.2, are very simple.

Corollary 5.2.

$$d_{n,1} = c_n \tag{5.6}$$

$$d_{n,2} = 2c_n - c_{n+1} \tag{5.7}$$

$$d_{n,3} = -2(n+1)c_n + 2c_{n+1}. \tag{5.8}$$

These representations for the correction terms using the Catalan numbers are conjectured by Baxter and Guttmann (1988). Expressions (5.6) and (5.7) were first proved by Bousquet-Mélou (1996) and then another derivation was given by the present authors (Inui and Katori 1996, see also, Katori *et al* 1997). The third one (5.8) was only announced in Inui and Katori (1996), its derivation is first given here.

6. Concluding remarks

Jensen and Guttmann (1995) calculated P_n as a power series of q up to $n = 39$. They found that (i) the correction terms $d_{n,l}$ can be written in the form

$$d_{n,l} = \sum_{k=1}^{\lfloor (l-1)/2 \rfloor} A_{l,k} \binom{n - m(l,k)}{k} c_{n-m(l,k)} + \sum_{k=1}^{2l-4} B_{l,k} c_{n-l+2+k} \tag{6.1}$$

for $3 \leqslant l \leqslant 15$ and $n \geqslant l-4$, where $m(l,k) = \max\{0, l-4-2k\}$ and (ii) for $l \leqslant 15$, the coefficients $A_{l,k}$ and $B_{l,k}$ are either integers or fractions with small (two or five) denominators.

Of course, corollary 5.1(ii) is consistent with these observations. It should be noted, however, that there is a gap between our theorems and their observations. In our representation (5.3), the number of terms, which are needed to express $d_{n,l}$ using the Catalan numbers, increases as n increases for a fixed l. On the other hand, (6.1) states that they have needed at most $2l-4$ terms for $d_{n,l}$ independently of n.

Now we want to introduce the following definition.

Definition 6.1. Let $\mathcal{C}(n-l, K; d)$ be the set of linear combinations of the Catalan numbers $\{c_{n'}\}$ in the form

$$\sum_{k=1}^{K} f_k(n) c_{n-l+k} \tag{6.2}$$

where $\{f_k(n)\}$ are the polynomials of n of at most degree d. If $\beta_{n,l} \in \mathcal{C}(n-l, K; h) \quad \forall n \geqslant l$ for a given l, we say that $\beta_{n,l}$ has the $\mathcal{C}(n-l, K; d)$-Catalan number representation (CNR).

Observation (6.1) implies that $d_{n,l}$ has the $\mathcal{C}(n-l+2, 2l-4, \lfloor (l-1)/2 \rfloor)$-CNR. Note that theorem 4.2 suggests the following conjecture.

Conjecture 6.2. For $s \geqslant 1$, $\{a_{n,m}^{(s)}\}$ has the $\mathcal{A}(n-s+1, [m-s+1, m+2s-1]; 2s)$-BNR.

If it is proved, we can conclude that $d_{n,l}$ has the $\mathcal{C}(n-l+1, 3(l-1), 2(l-1))$-CNR. Further investigation will be needed concerning the cancellation of terms, which occurs when we calculate $d_{n,m}$ from $\{a_{n,m}^{(s)}\}$ as shown in section 5.

Recently the extrapolation method has been extensively applied to many problems (Onody and Neves 1992, Essam *et al* 1996, Katori *et al* 1997, Jensen 1996, Jensen and Guttmann 1996b). We believe that the present paper shows a way to justify this method generally and to ensure the accuracy of series expansion data which give us fundamental information for unsolved problems.

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Appendix A. Proof of lemma 2.2

By (2.3) and (2.4), $f((n_2, m_2), (n_1, m_1); \Delta s)$ is the number of bond configurations satisfying many conditions. In this proof, we particularly concentrate on the condition about the number of closed bonds; exactly $(n_2 - n_1) - (|C_2| - C_1|) + \Delta s$ bonds are closed. In order to put emphasis on this condition, we write $r = (n_2 - n_1) - (m_2 - m_1) + \Delta s$ and define

$$\hat{f}((n_2, m_2), (n_1, m_1); r) = f((n_2, m_2), (n_1, m_1); \Delta s). \tag{A.1}$$

Consider a set $C_1 \in Y_{n_1}$ with $|C_1| = m_1$, $c(C_1) = 1$ and assume that $C_1 = \{x_1, x_1+2, \ldots, x_1'\}$ with $x_1' = x_1 + 2(m_1 - 1)$. Let $V^0_{[n_1,n_2]}(C_1)$ denote a trapezium $\{(x, y) \in \mathbb{Z}^2 : x + y = \text{even}, n_1 \leqslant y \leqslant n_2, x_1 - (y - n_1) \leqslant x \leqslant x_1' + (y - n_1)\}$, which is a subset of $V^0_{[n_1,n_2]}$. By using the definition of $A_n(A_{n_1} = B; \omega)$, we find that $\cup_{n=n_1}^{n_2} A_n(A_{n_1} = C_1; \omega) \subset V^0_{[n_1,n_2]}(C_1)$.

We assume that among r closed bonds r' bonds are in $V^0_{[n_1,n_2]}(C_1)$. Under this additional condition, we consider the bonds which are included in $V^0_{[n_1,n_2]}(C_1)$. The number of configurations of these bonds, which satisfy all the conditions given in (2.3) and (2.4), is given by $\hat{f}((m_1+(n_2-n_1), m_2), (m_1, m_1); r')$. Here we have used the fact that $V^0_{[n_1,n_2]}(C_1)$ is equivalent to $V^0_{[m_1, m_1+(n_2-n_1)]}$. Since the number of choices to select other $r - r'$ closed bonds from the set of bonds in $V^0_{[n_1,n_2]} \setminus V^0_{[n_1,n_2]}(C_1)$ is $\binom{2(n_2-n_1)(n_1-m_1)}{r-r'}$, we obtain the following formula.

$$\hat{f}((n_2, m_2), (n_1, m_1); r) = \sum_{r'=r_{\min}}^{r} \binom{2(n_2 - n_1)(n_1 - m_1)}{r - r'} \times \hat{f}((m_1 + (n_2 - n_1), m_2), (m_1, m_1); r') \tag{A.2}$$

where $r_{\min}$ is the possible minimum value of r', which will be determined below. The important point is that $\hat{f}((m_1 + (n_2 - n_1), m_2), (m_1, m_1); r)$ depends on the height of the trapezium $n_2 - n_1$ but is independent of the absolute value of n_1. Formula (A.2) states that, if we regard $\hat{f}((n_2, m_2), (n_1, m_1); r)$ as a polynomial of n_1 for fixed $n_2 - n_1$, the degree is at most $r - r_{\min}$.

Now the problem is reduced to determining $r_{\min}$ as a function of $\Delta n = n_2 - n_1$ and $\Delta m = m_2 - m_1$. The precise definition of $r_{\min}$ is the following.

$$r_{\min}(\Delta n, \Delta m) = \text{minimum number of the closed bonds in } \omega \in \mathcal{B}_{n_1,n_2}, \text{ such that}$$
$$A_{n_2}(A_{n_1} = C_1; \omega) = C_2, |C_1| = m_1, c(C_1) = 1, |C_2| = m_2, c(C_2) = 1$$
$$\text{and } c(A_n(A_{n_1} = C_1; \omega)) \geqslant 2 \ \forall n \in \{n_1 + 1, \ldots, n_2 - 1\}. \tag{A.3}$$

First we consider $r^*_{\min}(\Delta n, \Delta m)$ which is defined in a similar way to (A.3), but in which the condition $c(A_n(A_{n_1} = C_1; \omega)) \geqslant 2 \ \forall n \in \{n_1 + 1, \ldots, n_2 - 1\}$ is replaced by $|A_n(A_{n_1} = C_1; \omega)| \geqslant 3$ and $c(A_n(A_{n_1} = C_1; \omega)) = 1 \ \forall n \in \{n_1 + 1, \ldots, n_2 - 1\}$. Some consideration leads us to that, if $\Delta n \geqslant 2$,

$$r^*_{\min}(\Delta n, \Delta m) = \begin{cases} \Delta n - \Delta m & \text{if } -\Delta n < \Delta m \leqslant \Delta n \\ -2\Delta m & \text{if } \Delta m \leqslant -\Delta n. \end{cases} \tag{A.4}$$

It is easy to find that

$$r_{\min}(\Delta n, \Delta m) = r^*_{\min}(\Delta n, \Delta m) + \begin{cases} \Delta n & \text{if } -\Delta n < \Delta m \leqslant \Delta n \\ \Delta n - 1 & \text{if } \Delta m \leqslant -\Delta n. \end{cases} \tag{A.5}$$

We also find that

$$r_{\min}(\Delta n = 1, \Delta m) = \begin{cases} 0 & \text{if } \Delta m = 1 \\ 1 & \text{if } \Delta m = 0 \\ -2\Delta m & \text{if } \Delta m < 0. \end{cases} \tag{A.6}$$

Since $r = \Delta n - \Delta m + \Delta s$, we obtain the results.

$$r - r_{\min} = \begin{cases} \Delta s - \Delta n & \text{if } \Delta n \geqslant 2 \text{ and } -\Delta n < \Delta m \leqslant \Delta n \\ \Delta s + \Delta m + 1 & \text{if } \Delta n \geqslant 2 \text{ and } \Delta m \leqslant -\Delta n \\ \Delta s & \text{if } \Delta n = 1 \text{ and } \Delta m = 0 \text{ or } 1 \\ \Delta s + \Delta m + 1 & \text{if } \Delta n = 1 \text{ and } \Delta m < 0 \end{cases} \tag{A.7}$$

which gives $r - r_{\min} \leqslant \Delta s$. Therefore, $f((n+1, m), (n-k, m'); s - s')$ is the polynomials of n of at most degree $s - s'$.

For $g((n, m), (n - k - 1, m'); s - s')$, we can perform the same procedure to evaluate the degree with respect to n. In this case, $r_{\min}$ can be greater than (A.3), since the condition $c(C_2) = 1$ is replaced by $c(C_2) \geqslant 2$ for $g(\cdot)$ as defined by (2.5). It follows that $r - r_{\min}$ can be reduced. The highest degree is bounded from above by $s - s'$ anyway. This completes the proof. □

Appendix B. Proof of lemma 3.1

Assume that a double series $\{\beta_{n,m}\}$ is defined for $n \geqslant n_0 \geqslant 1$ and $m \geqslant 1$. Let the value $D(\beta_{n,m})$ be $I_{n,m}$ for each n, m. First we extend this series for $m \leqslant 0$. We assume the following asymmetry with respect to m;

$$\beta_{n,m} = -\beta_{n,-m} \qquad \text{for } m \leqslant 0. \tag{B.1}$$

We also assume $I_{n,m} = -I_{n,-m}$ for $m \leqslant 0$ and have

$$D(\beta_{n,m}) = I_{n,m} \qquad \text{for } n \geqslant n_0 \qquad -\infty < m < \infty. \tag{B.2}$$

The generating function for $\{\beta_{n,m}\}$ is introduced as

$$\Psi(x, y) = \sum_{n=n_0}^{\infty} \sum_{m=-\infty}^{\infty} x^n y^m \beta_{n,m}. \tag{B.3}$$

If we regard (B.2) as a difference equation, it gives an equation for $\Psi(x, y)$,

$$\Psi(x, y) = K(x, y)[x^{n_0-1}\Psi(y) + I(x, y)] \tag{B.4}$$

with

$$K(x, y) = -\frac{xy}{x(y+1)^2 - y} \tag{B.5}$$

$$\Psi(y) = \sum_{m=-\infty}^{\infty} y^m \beta_{n_0,m} \tag{B.6}$$

$$I(x, y) = \sum_{n=n_0}^{\infty} \sum_{m=-\infty}^{\infty} x^n y^m I_{n,m}. \tag{B.7}$$

In order to derive it, we have used (B.1), which gives that $\beta_{n,0} = 0$ and $\sum_n x^n(\beta_{n,-1} + \beta_{n,1}) = 0$.

First we notice that

$$K(x, y) = \left(y - \frac{1}{y}\right)^{-1} \Phi(x, y) \tag{B.8}$$

with

$$\Phi(x, y) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} x^n y^m \alpha_{n,m} \tag{B.9}$$

which is the generating function of the ballot numbers. This observation leads to

$$K(x, y)x^{n_0-1}\Psi(y) = \sum_{n=n_0}^{\infty} \sum_{m=-\infty}^{\infty} x^n y^m \sum_{t=1}^{\infty} \alpha_{n-n_0+1,t} \sum_{k=1}^{t} \beta_{n_0,m-t+2k-1}. \tag{B.10}$$

Next we see that

$$K(x, y) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} x^n y^m \binom{2(n-1)}{(n-1)+m}. \tag{B.11}$$

This gives

$$K(x, y)I(x, y) = \sum_{n=n_0+1}^{\infty} \sum_{m=-\infty}^{\infty} x^n y^m L(n, m) \tag{B.12}$$

with

$$\begin{aligned} L(n, m) &= \sum_{l=1}^{n-n_0} \sum_{w=-\infty}^{\infty} \binom{2(n-n_0-l)}{(n-n_0-l)+m-w} I_{l+n_0-1,w} \\ &= \sum_{l=1}^{n-n_0} \sum_{w=1}^{\infty} \alpha_{n-n_0-l+1,m,w} I_{l+n_0-1,w} \end{aligned} \tag{B.13}$$

where the property $I_{l,-w} = -I_{l,w}$ has been used. Comparing coefficients of terms with $x^n y^m$ in (B.4), we obtain for $n \geqslant n_0$

$$\beta_{n,m} = \sum_{t=1}^{\infty} \alpha_{n-n_0+1,t} \sum_{k=1}^{t} \beta_{n_0,m-t+2k-1} + \sum_{l=1}^{n-n_0} \sum_{w=1}^{\infty} \alpha_{n-n_0-l+1,m,w} D(\beta_{l+n_0-1,w}) \tag{B.14}$$

where (B.2) was used.

Now we put a condition for $\{\beta_{n,m}\}$;

$$\beta_{n,m} = 0 \qquad \text{for } |m| > n. \tag{B.15}$$

Under this condition and (B.1), the first term on the RHS of (B.14) becomes that of (3.8).

Appendix C. Proof of lemma 3.3

In the previous paper (Inui and Katori 1996), the following useful lemma was proved.

Lemma C.1. Define

$$a(x) = \frac{1}{2x}\left\{1 - 2x - \sqrt{1-4x}\right\}. \tag{C.1}$$

Then

$$a(x)^m = \sum_{n=1}^{\infty} \alpha_{n,m} x^n \qquad \text{for } m \geqslant 1. \tag{C.2}$$

Let a differential operator, $\mathcal{D}_x$, be

$$\mathcal{D}_x = x^2 \frac{\partial}{\partial x}. \tag{C.3}$$

We see

$$\mathcal{D}_x^k a(x)^m = \sum_{n=1}^{\infty} (n)_k \alpha_{n,m} x^{n+k} \qquad \text{for } k \geqslant 0 \tag{C.4}$$

where $(n)_0 = 1$, $(n)_k = n(n+1)\ldots(n+k-1)$ for $k \geqslant 1$. On the other hand, it is easy to confirm that

$$\frac{a(x)^{m-\gamma+1}}{1-a(x)^2} = \sum_{n=1}^{\infty} \binom{2n}{n+m-\gamma} x^{n+1} \tag{C.5}$$

if $m - \gamma \geqslant 0$. Therefore, the following identity holds.

Lemma C.2. For $m - \gamma \geqslant 0$, $k \geqslant 0$ and $\delta \geqslant 1$,

$$\begin{aligned} B_\gamma^{(k)}(x) &\equiv \frac{a(x)^{m-\gamma+1}}{1-a(x)^2} \mathcal{D}_x^k a(x)^\delta \\ &= \sum_{n=2}^{\infty} x^{n+k} \sum_{l=1}^{n-1} \binom{2(n-l-1)}{(n-l-1)+m-\gamma} (l)_k \alpha_{l,\delta}. \end{aligned} \tag{C.6}$$

First we consider the case $k = 0$.

$$\begin{aligned} B_\gamma^{(0)}(x) &= \frac{a(x)^{m+\delta-\gamma+1}}{1-a(x)^2} \\ &= \sum_{k=0}^{\infty} a(x)^{m+\delta-\gamma+2k+1} \\ &= \sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} \alpha_{n,m+\delta-\gamma+2k+1} \end{aligned} \tag{C.7}$$

in which we used (C.2) for the last equality. Thus we have

$$\sum_{l=1}^{n-1} \binom{2(n-l-1)}{(n-l-1)+m-\gamma} \alpha_{l,\delta} = \sum_{k=0}^{\infty} \alpha_{n,m+\delta-\gamma+2k+1}. \tag{C.8}$$

Obtain an equation by changing γ to $-\gamma$ in (C.8) and subtract it from (C.8). By (3.1), we have the identity

$$\sum_{l=1}^{n-1} \alpha_{n-l,m,\gamma} \alpha_{l,\delta} = \sum_{k=0}^{\gamma-1} \alpha_{n,m+\delta-\gamma+2k+1}. \tag{C.9}$$

Next we consider the case $k = 1$. By (C.1) we have

$$\mathcal{D}_x a(x)^m = \frac{m a(x)^{m+1}}{1-a(x)^2}. \tag{C.10}$$

Therefore, we obtain

$$\begin{aligned} B_\gamma^{(1)}(x) &= \frac{\delta a(x)^{m+\delta-\gamma+2}}{(1-a(x)^2)^2} \\ &= \sum_{k=0}^{\infty} \frac{\delta a(x)^{m+\delta-\gamma+2k+2}}{1-a(x)^2} \\ &= \sum_{k=0}^{\infty} \frac{\delta}{m+\delta-\gamma+2k+1} \mathcal{D}_x (a(x)^{m+\delta-\gamma+2k+1}) \\ &= \sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} \frac{\delta (n)_1}{m+\delta-\gamma+2k+1} \alpha_{n,m+\delta-\gamma+2k+1} \end{aligned} \tag{C.11}$$

in which we used (C.4) for the last equality. Following the same procedure as for the case $k = 0$, we find

$$\sum_{l=1}^{n-1} \alpha_{n-l,m,\gamma} (l)_1 \alpha_{l,\delta} = \sum_{k=0}^{\gamma-1} \frac{\delta (n)_1}{m+\delta-\gamma+2k+1} \alpha_{n,m+\delta-\gamma+2k+1}. \tag{C.12}$$

When $k = 2$, using

$$\mathcal{D}_x^2 a(x)^m = m(m-1) \frac{a(x)^{m+2}}{(1-a(x)^2)^2} + 2m \frac{a(x)^{m+2}}{(1-a(x)^2)^3} \tag{C.13}$$

we obtain

$$\begin{aligned} B_\gamma^{(2)}(x) = \sum_{k=0}^{\infty} & \frac{\delta}{m-\gamma+\delta+2k+1} \mathcal{D}_x^2 (a(x)^{m-\gamma+\delta+2k+1}) \\ & - \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{\delta(m-\gamma+2k+1)}{m-\gamma+\delta+2(k+p)+2} \mathcal{D}_x (a(x)^{m-\gamma+\delta+2(k+p)+2}). \end{aligned} \tag{C.14}$$

This gives an identity similar to (C.12).

Since (3.1) gives

$$\alpha_{n,m,-k} = -\alpha_{n,m,k} \tag{C.15}$$

the above identities are generalized as follows. Remark that $\alpha_{n,m,0} = 0$ by (C.15).

Lemma C.3. Assume that $\delta \geqslant 1$. If $m-\gamma \geqslant 0$ and $\gamma \geqslant 1$ or $m-\gamma \leqslant 0$ and $\gamma \leqslant -1$, then

$$\sum_{l=1}^{n-1} \alpha_{n-l,m,\gamma}\alpha_{l,\delta} = \mathrm{sgn}(\gamma) \sum_{k=0}^{|\gamma|-1} \alpha_{n,|m-\gamma|+\delta+2k+1} \tag{C.16}$$

$$\sum_{l=1}^{n-1} \alpha_{n-l,m,\gamma}(l)_1\alpha_{l,\delta} = \mathrm{sgn}(\gamma) \sum_{k=0}^{|\gamma|-1} \frac{\delta(n)_1}{|m-\gamma|+\delta+2k+1}\alpha_{n,|m-\gamma|+\delta+2k+1} \tag{C.17}$$

$$\sum_{l=1}^{n-1} \alpha_{n-l,m,\gamma}(l)_2\alpha_{l,\delta} = \mathrm{sgn}(\gamma)\left\{ \sum_{k=0}^{|\gamma|-1} \frac{\delta(n)_2}{|m-\gamma|+\delta+2k+1}\alpha_{n,|m-\gamma|+\delta+2k+1} \right.$$
$$\left. - \sum_{k=0}^{|\gamma|-1}\sum_{p\geqslant 0} \frac{\delta(|m-\gamma|+2k+1)(n+1)_1}{|m-\gamma|+\delta+2(k+p)+2}\alpha_{n+1,|m-\gamma|+\delta+2(k+p)+2}\right\} \tag{C.18}$$

where $\mathrm{sgn}(\gamma) = \gamma/|\gamma|$.

In general

$$\mathcal{D}_x^k a(x)^m = a(x)^{m+k} \sum_{l=k}^{2k-1} U_{k,l}(m) \frac{1}{(1-a(x)^2)^l} \tag{C.19}$$

for $k \geqslant 1$. Here $\{U_{k,l}(m)\}$ are polynomials of m, which are determined by the following iteration

$$\begin{aligned} &U_{k+1,k+1}(m) = (m-k)U_{k,k}(m) \\ &U_{k+1,l}(m) = (m+k-2l+2)U_{k,l-1}(m) + 2(l-2)U_{k,l-2}(m) \\ &\qquad \text{for } k+2 \leqslant l \leqslant 2k \\ &U_{k+1,2(k+1)-1}(m) = 2(2k-1)U_{k,2k-1}(m) \end{aligned} \tag{C.20}$$

with the condition $U_{1,l}(m) = m\delta_{l,1}$. Using (C.19) successively, we have

$$B_\gamma^{(k)}(x) = \sum_{q=0}^{k-1}\sum_{p_0\geqslant 0}\sum_{p_1\geqslant 0}\dots\sum_{p_q\geqslant 0} R\Bigg(\delta, m-\gamma+\delta+2p_0+1, m-\gamma+\delta+2(p_0+p_1)+2,$$
$$\dots, m-\gamma+\delta+2\sum_{i=0}^{q} p_i + q + 1\Bigg)\mathcal{D}_x^{k-q}\left(a(x)^{m-\gamma+\delta+2\sum_{i=0}^q p_i+q+1}\right) \tag{C.21}$$

where $\{R(x_1, x_2, \dots, x_{q+2})\}$ are rational functions of $U_{k_1,l_1}(x_1), U_{k_2,l_2}(x_2), \dots, U_{k_{q+2},l_{q+2}}$ (x_{q+2}) with appropriate $\{k_i, l_i\}$'s.

It follows that

$$B_\gamma^{(k)}(x) - B_{-\gamma}^{(k)}(x) = \sum_{n=2}^{\infty} x^{n+k} \sum_{q=0}^{k-1}\sum_{p_0=0}^{\gamma-1}\sum_{p_1\geqslant 0}$$
$$\dots\sum_{p_q\geqslant 0}\tilde{R}(m,\gamma,\delta,p_0,p_1,\dots,p_q)(n+q)_{k-q}\alpha_{n+q,m-\gamma+\delta+2\sum_{i=0}^q p_i+q+1} \tag{C.22}$$

with appropriate functions $\{\tilde{R}\}$. On the other hand, lemma C.2 gives

$$B_\gamma^{(k)}(x) - B_{-\gamma}^{(k)}(x) = \sum_{n=2}^{\infty} x^{n+k} \sum_{l=1}^{n-1} \alpha_{n-l,m,\gamma}(l)_k\alpha_{l,\delta}. \tag{C.23}$$

Since $\alpha_{n+q,m'} \in \mathcal{A}(n;0)$ for $q \geqslant 0$ by (3.6), we can conclude that

$$\sum_{l=1}^{n-1} \alpha_{n-l,m,\gamma}(l)_k\alpha_{l,\delta} = \sum_{p\geqslant 0} f_p(n;m,\gamma,\delta;k)\alpha_{n,p}. \tag{C.24}$$

Here $f_p(n;m,\gamma,\delta;k)$ are the polynomials of n of at most degree k, whose coefficients depend on the values of m, γ, δ. Thus, lemma 3.3 has been proved.

Appendix D. Proof of lemma 4.4

Define

$$\Phi^+(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_{n,m}x^n y^m$$
$$\Phi^-(x, y) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{-1}\alpha_{n,m}x^n y^m \tag{D.1}$$

lemma C.1 gives that

$$\Phi^+(x, y) = \frac{a(x)y}{1-a(x)y}$$
$$\Phi^-(x, y) = -\Phi^+(x, y^{-1}) = -\frac{a(x)}{y-a(x)}. \tag{D.2}$$

It is easy to confirm that

$$\mathcal{D}_x\Phi^{\pm}(x, y) = K(x, y)[\Phi^{\pm}(x, y) \mp \mathcal{D}_x a(x)] \tag{D.3}$$

where $K(x, y)$ and $\mathcal{D}_x$ are defined as (B.5) and (C.3), respectively.

Let

$$\Phi_t(x, y) = y^{-t}\Phi^+(x, y) + y^t\Phi^-(x, y). \tag{D.4}$$

Then, (D.3) gives

$$\mathcal{D}_x\Phi_t(x, y) = K(x, y)\left[\Phi_t(x, y) + \left(y^t - \frac{1}{y^t}\right)\mathcal{D}_x a(x)\right]. \tag{D.5}$$

We find that $K(x, y)$ has the following property.

$$\mathcal{D}_x K(x, y) = (K(x, y))^2. \tag{D.6}$$

Only by using (D.5) and (D.6), can we prove the following identities.

Lemma D.1. For any $k \geqslant 0$,

$$(k+1)K(x, y)\mathcal{D}_x^k\Phi_t(x, y) = \mathcal{D}_x^{k+1}\Phi_t(x, y) - \left(y^t - \frac{1}{y^t}\right)K(x, y)\mathcal{D}_x^{k+1}a(x). \tag{D.7}$$

By definition and (B.11), we find that

$$K(x, y)\mathcal{D}_x^k\Phi_t(x, y) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}x^{n+k+1}y^m\sum_{l=1}^{n}\sum_{w=-t+1}^{\infty}\alpha_{n+1-l,m,w}(l)_k\alpha_{l,w+t} \tag{D.8}$$

$$\mathcal{D}_x^{k+1}\Phi_t(x, y) = \sum_{n=1}^{\infty}x^{n+k+1}\left\{\sum_{m=-\infty}^{t-1}y^m(n)_{k+1}\alpha_{n,m-t} + \sum_{m=-t+1}^{\infty}y^m(n)_{k+1}\alpha_{m+t}\right\} \tag{D.9}$$

and

$$\left(y^t - \frac{1}{y^t}\right)K(x, y)\mathcal{D}^{k+1}a(x) = \sum_{n=2}^{\infty}\sum_{m=-\infty}^{\infty}x^{n+k+1}y^m\sum_{l=1}^{n-1}\alpha_{n-l,m,t}(l)_{k+1}\alpha_{l,1}. \tag{D.10}$$

Lemma 4.4(i) is derived by comparing the coefficients of terms with $x^n y^m$ in (D.7).

Setting $\delta = 1$ in lemma C.3 gives lemma 4.4(ii).

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